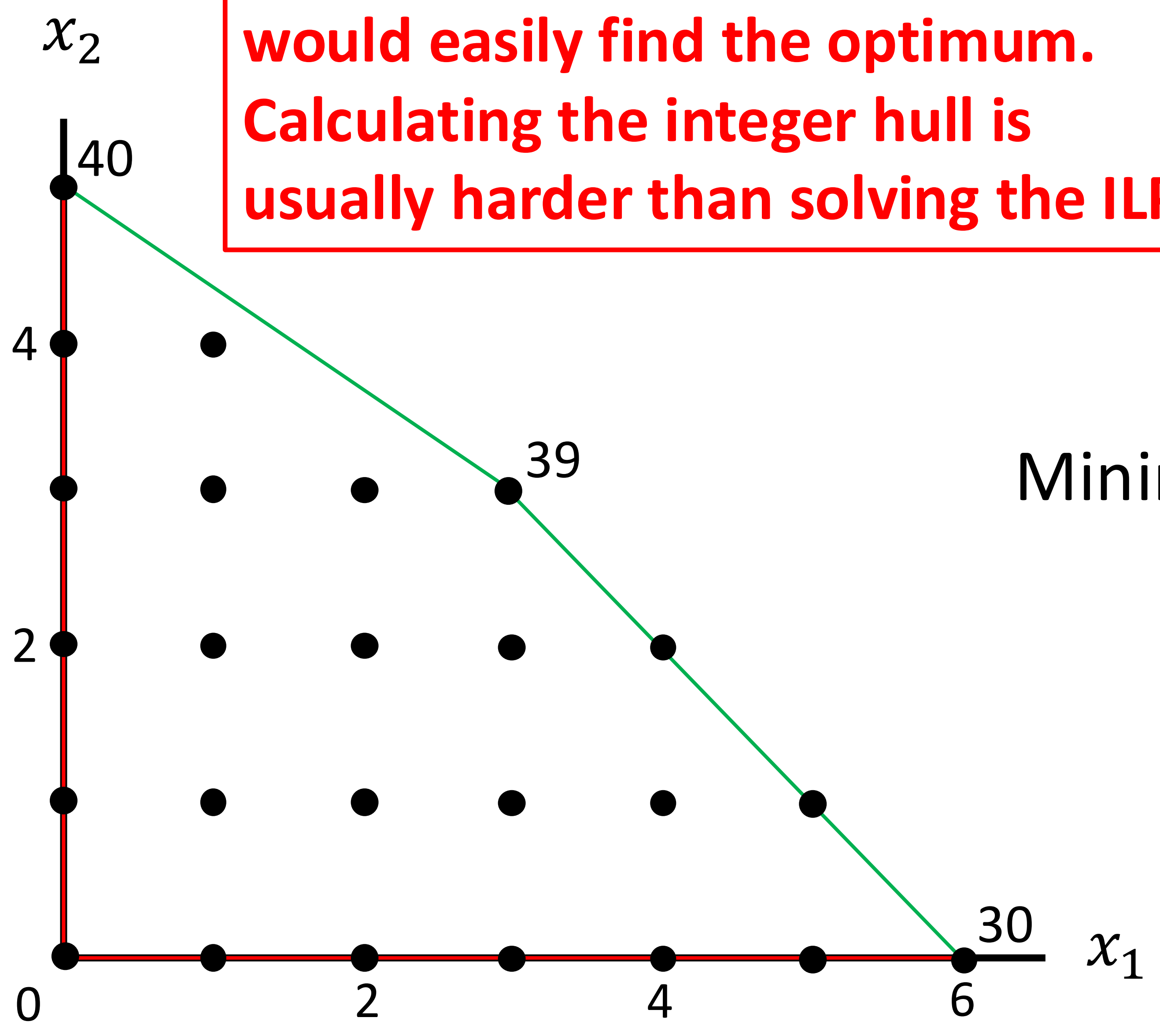
Integer Linear Programming
CSCI 532



If you had the integer hull, Simplex would easily find the optimum. Calculating the integer hull is usually harder than solving the ILP.

$x_1, x_2 \in \mathbb{N}$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$5x_1 + 9x_2 \leq 45$

$x_1, x_2 \geq 0$

Minimal convex hull (integer hull):

- Convex.
- local optimum = global optimum.
- $O\left(n^{\lfloor d/2 \rfloor}\right)$ faces,

  $n = \#$ points

  $d = \#$ dimensions

ILP vs LP

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Optimal:

$x_1 = 0$
$x_2 = 1$
$x_3 = 1$
$x_4 = 1$
$x_5 = 0$

Objective = 3

$x_i \in [0,1]$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

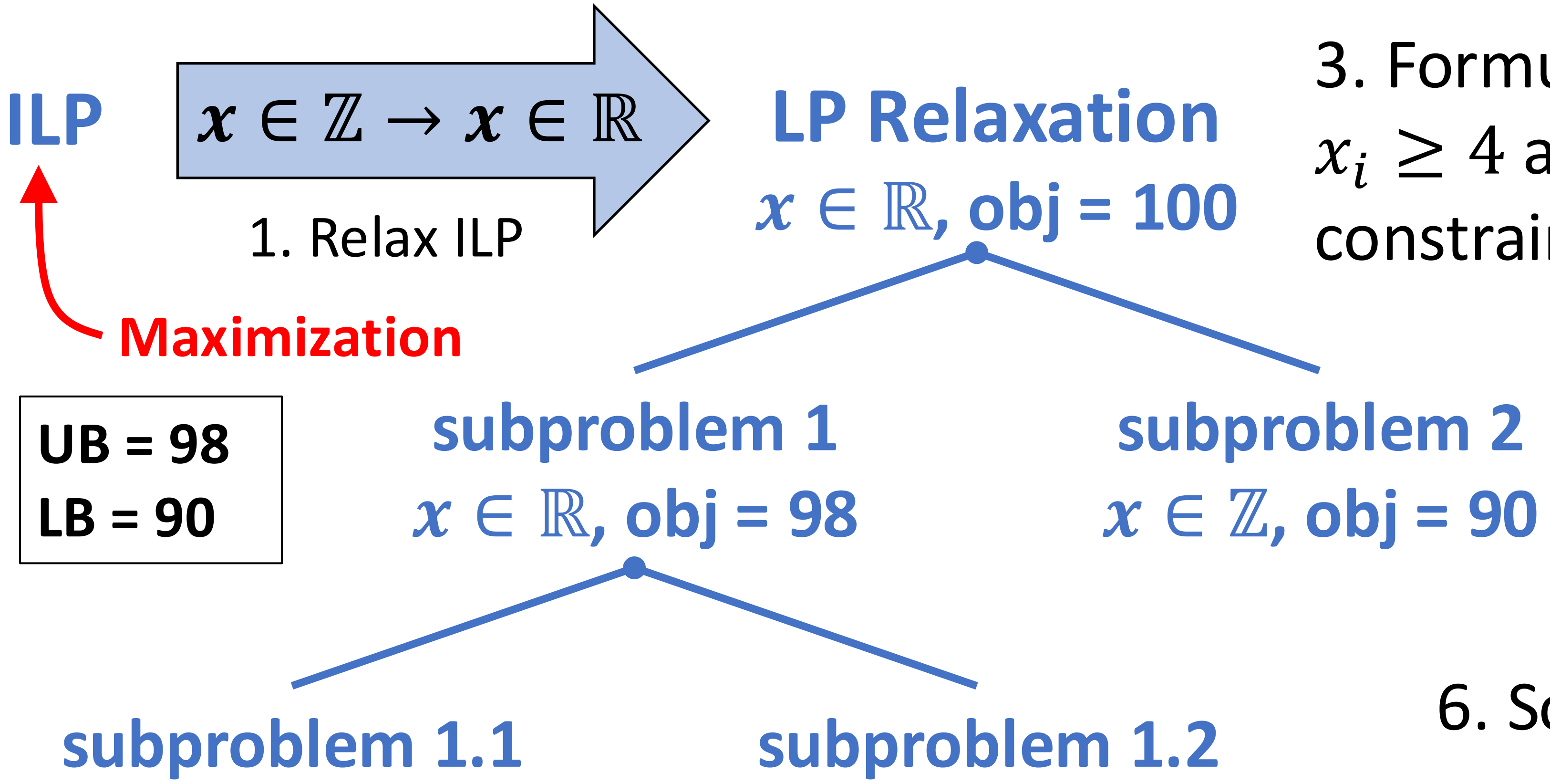
Optimal:

$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$

Objective = 2.5

Since the LP has **more options** to reduce the objective value, $OPT_{LP} \leq OPT_{ILP}$ (for a minimization problem). If the minimum objective value comes from an integer solution, a plain LP solver (e.g., Simplex) will find it.

Solving ILPs



2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$x \in \mathbb{R}$, **obj = 100**

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

UB = ?

LB = ?

subproblem 1

$x \in \mathbb{R}$, **obj = 98**

subproblem 2

$x \in \mathbb{Z}$, **obj = 90**

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

subproblem 1.1

$x \in \mathbb{R}$, **obj = 88**

subproblem 1.2

$x \in \mathbb{R}$, **obj = 97**

What happens?

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$x \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

UB = 97
LB = 90

subproblem 1
$x \in \mathbb{R}$, obj = 98

subproblem 2
$x \in \mathbb{Z}$, obj = 90

~~**subproblem 1.1**~~
$x \in \mathbb{R}$, obj = 88

subproblem 1.2
$x \in \mathbb{R}$, obj = 97

What happens?

subproblem 1.1 is a dead end.
Prune it and keep going.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, obj = 100

subproblem 1

$\boldsymbol{x} \in \mathbb{R}$, obj = 98

subproblem 2

$\boldsymbol{x} \in \mathbb{Z}$, obj = 90

subproblem 1.1

$\boldsymbol{x} \in \mathbb{R}$, obj = 88

subproblem 1.2

$\boldsymbol{x} \in \mathbb{R}$, obj = 85

UB = ?

LB = ?

What happens?

Solving ILPs

ILP

1. Relax ILP: $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

Maximization

LP Relaxation
$x \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

subproblem 1
$x \in \mathbb{R}$, obj = 98

subproblem 2
$x \in \mathbb{Z}$, obj = 90

Optimal

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

~~**subproblem 1.1**~~
$x \in \mathbb{R}$, obj = 88

~~**subproblem 1.2**~~
$x \in \mathbb{R}$, obj = 85

UB = 90
LB = 90

What happens?

Solving ILPs

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

Maximization

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

LP Relaxation
$\boldsymbol{x} \in \mathbb{R}$**, obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

UB = 97
LB = 90

subproblem 1
$\boldsymbol{x} \in \mathbb{R}$**, obj = 98**

subproblem 2
$\boldsymbol{x} \in \mathbb{Z}$**, obj = 90**

5. Formulate two new LPs that split the subproblem on some non-integer variable.

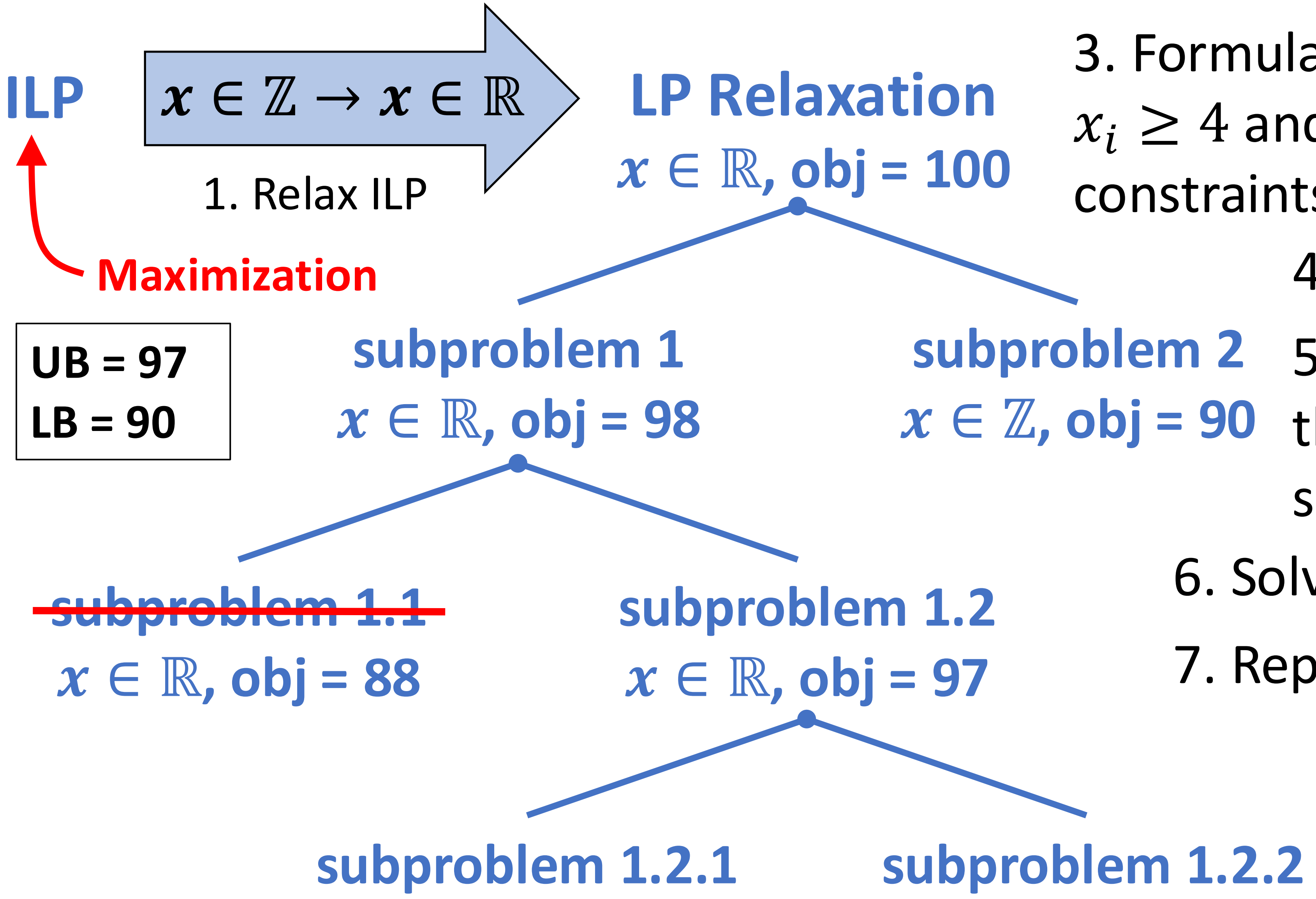
~~**subproblem 1.1**~~
$\boldsymbol{x} \in \mathbb{R}$**, obj = 88**

subproblem 1.2
$\boldsymbol{x} \in \mathbb{R}$**, obj = 97**

6. Solve the subproblem LPs.

7. Repeat.

Solving ILPs



2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

UB = ?

LB = ?

LP Relaxation

$x \in \mathbb{R}$, **obj = 100**

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

subproblem 1

$x \in \mathbb{R}$, **obj = 98**

subproblem 2

$x \in \mathbb{Z}$, **obj = 90**

5. Formulate two new LPs that split the subproblem on some non-integer variable.

~~**subproblem 1.1**~~

$x \in \mathbb{R}$, **obj = 88**

subproblem 1.2

$x \in \mathbb{R}$, **obj = 97**

6. Solve the subproblem LPs.

7. Repeat.

subproblem 1.2.1

$x \in \mathbb{R}$, **obj = 94**

subproblem 1.2.2

$x \in \mathbb{Z}$, **obj = 95**

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

ILP — $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ — **LP Relaxation**

1. Relax ILP

Maximization

UB = 95
LB = 95

LP Relaxation
$x \in \mathbb{R}$, obj = 100

subproblem 1
$x \in \mathbb{R}$, obj = 98

subproblem 2
$x \in \mathbb{Z}$, obj = 90

~~subproblem 1.1~~
$x \in \mathbb{R}$, obj = 88

subproblem 1.2
$x \in \mathbb{R}$, obj = 97

~~subproblem 1.2.1~~
$x \in \mathbb{R}$, obj = 94

subproblem 1.2.2
$x \in \mathbb{Z}$, obj = 95 ← **Optimal**

What happens?

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$x \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

UB = ?

LB = ?

subproblem 1
$x \in \mathbb{R}$, obj = 98

subproblem 2
$x \in \mathbb{Z}$, obj = 90

~~subproblem 1.1~~
$x \in \mathbb{R}$, obj = 88

subproblem 1.2
$x \in \mathbb{R}$, obj = 97

subproblem 1.2.1
$x \in \mathbb{R}$, obj = 85

subproblem 1.2.2
$x \in \mathbb{Z}$, obj = 89

What happens?

Solving ILPs

ILP → 1. Relax ILP: $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ → **LP Relaxation**

Maximization

UB = 90
LB = 90

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

LP Relaxation
$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

- **subproblem 1**
  $\boldsymbol{x} \in \mathbb{R}$, **obj = 98**
  - ~~**subproblem 1.1**~~
    $\boldsymbol{x} \in \mathbb{R}$, **obj = 88**
  - **subproblem 1.2**
    $\boldsymbol{x} \in \mathbb{R}$, **obj = 97**
    - ~~**subproblem 1.2.1**~~
      $\boldsymbol{x} \in \mathbb{R}$, **obj = 85**
    - **subproblem 1.2.2**
      $\boldsymbol{x} \in \mathbb{Z}$, **obj = 89**
- **subproblem 2**
  $\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**
  Optimal

What happens?

Solving ILPs

ILP — $x \in \mathbb{Z} \rightarrow x \in \mathbb{R}$ — **LP Relaxation**

1. Relax ILP

Maximization

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

LP Relaxation
$x \in \mathbb{R}$, **obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

UB = ?
LB = ?

subproblem 1
$x \in \mathbb{R}$, **obj = 98**

subproblem 2
$x \in \mathbb{Z}$, **obj = 90**

5. Formulate two new LPs that split the subproblem on some non-integer variable.

~~**subproblem 1.1**~~
$x \in \mathbb{R}$, **obj = 88**

subproblem 1.2
$x \in \mathbb{R}$, **obj = 97**

6. Solve the subproblem LPs.

7. Repeat.

subproblem 1.2.1
$x \in \mathbb{R}$, **obj = 89**

subproblem 1.2.2
$x \in \mathbb{R}$, **obj = 88**

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

ILP — $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ — **LP Relaxation**

1. Relax ILP

Maximization

UB = 90
LB = 90

LP Relaxation
$x \in \mathbb{R}$, **obj = 100**

subproblem 1
$x \in \mathbb{R}$, **obj = 98**

subproblem 2
$x \in \mathbb{Z}$, **obj = 90**

Optimal

~~**subproblem 1.1**~~
$x \in \mathbb{R}$, **obj = 88**

subproblem 1.2
$x \in \mathbb{R}$, **obj = 97**

~~**subproblem 1.2.1**~~
$x \in \mathbb{R}$, **obj = 89**

~~**subproblem 1.2.2**~~
$x \in \mathbb{R}$, **obj = 88**

What happens?

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation
$x \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

UB = ?
LB = ?

subproblem 1
$x \in \mathbb{R}$, obj = 98

subproblem 2
$x \in \mathbb{Z}$, obj = 90

~~subproblem 1.1~~
$x \in \mathbb{R}$, obj = 88

subproblem 1.2
$x \in \mathbb{R}$, obj = 97

subproblem 1.2.1
$x \in \mathbb{R}$, obj = 94

subproblem 1.2.2
$x \in \mathbb{Z}$, obj = 88

What happens?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

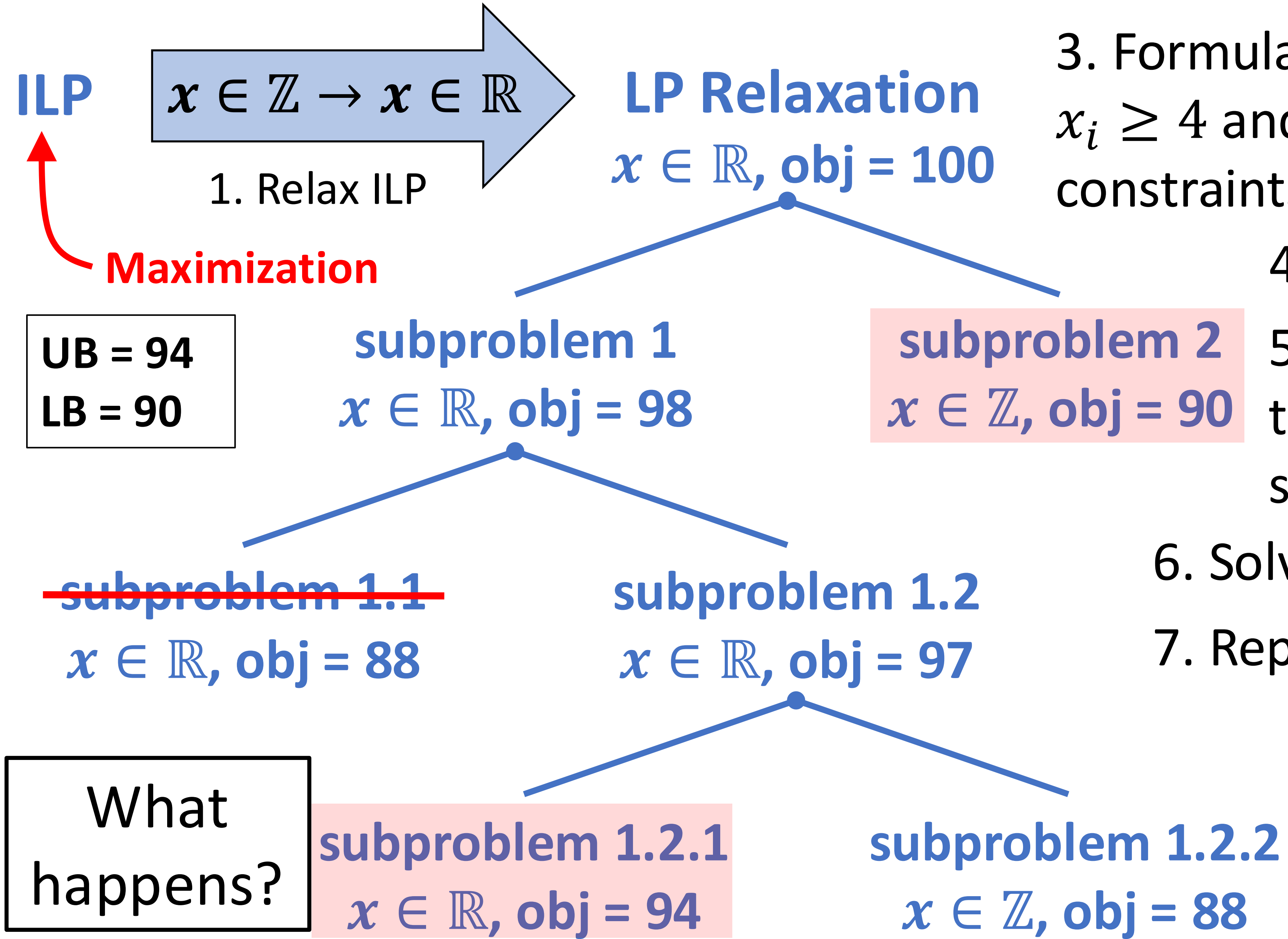
3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.



Need to continue. There could be an integer solution buried in subproblem 1.2.1 with a better objective value.

Solving ILPs

ILP ← **Maximization**

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

LP Relaxation
$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

UB = ?
LB = ?

subproblem 1
$\boldsymbol{x} \in \mathbb{R}$, **obj = 98**

subproblem 2
$\boldsymbol{x} \in \mathbb{Z}$, **obj = 90**

~~**subproblem 1.1**~~
$\boldsymbol{x} \in \mathbb{R}$, **obj = 88**

subproblem 1.2
$\boldsymbol{x} \in \mathbb{R}$, **obj = 97**

subproblem 1.2.1
$\boldsymbol{x} \in \mathbb{R}$, **obj = 94**

subproblem 1.2.2
$\boldsymbol{x} \in \mathbb{Z}$, **obj = 92**

What happens?

Solving ILPs

ILP

$\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$

1. Relax ILP

Maximization

LP Relaxation

$\boldsymbol{x} \in \mathbb{R}$, obj = 100

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.

UB = 94
LB = 92

subproblem 1
$\boldsymbol{x} \in \mathbb{R}$, obj = 98

subproblem 2
$\boldsymbol{x} \in \mathbb{Z}$, obj = 90

~~**subproblem 1.1**~~
$\boldsymbol{x} \in \mathbb{R}$, obj = 88

subproblem 1.2
$\boldsymbol{x} \in \mathbb{R}$, obj = 97

subproblem 1.2.1
$\boldsymbol{x} \in \mathbb{R}$, obj = 94

subproblem 1.2.2
$\boldsymbol{x} \in \mathbb{Z}$, obj = 92

What happens?

Need to continue. There could be an integer solution buried in subproblem 1.2.1 with a better objective value.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

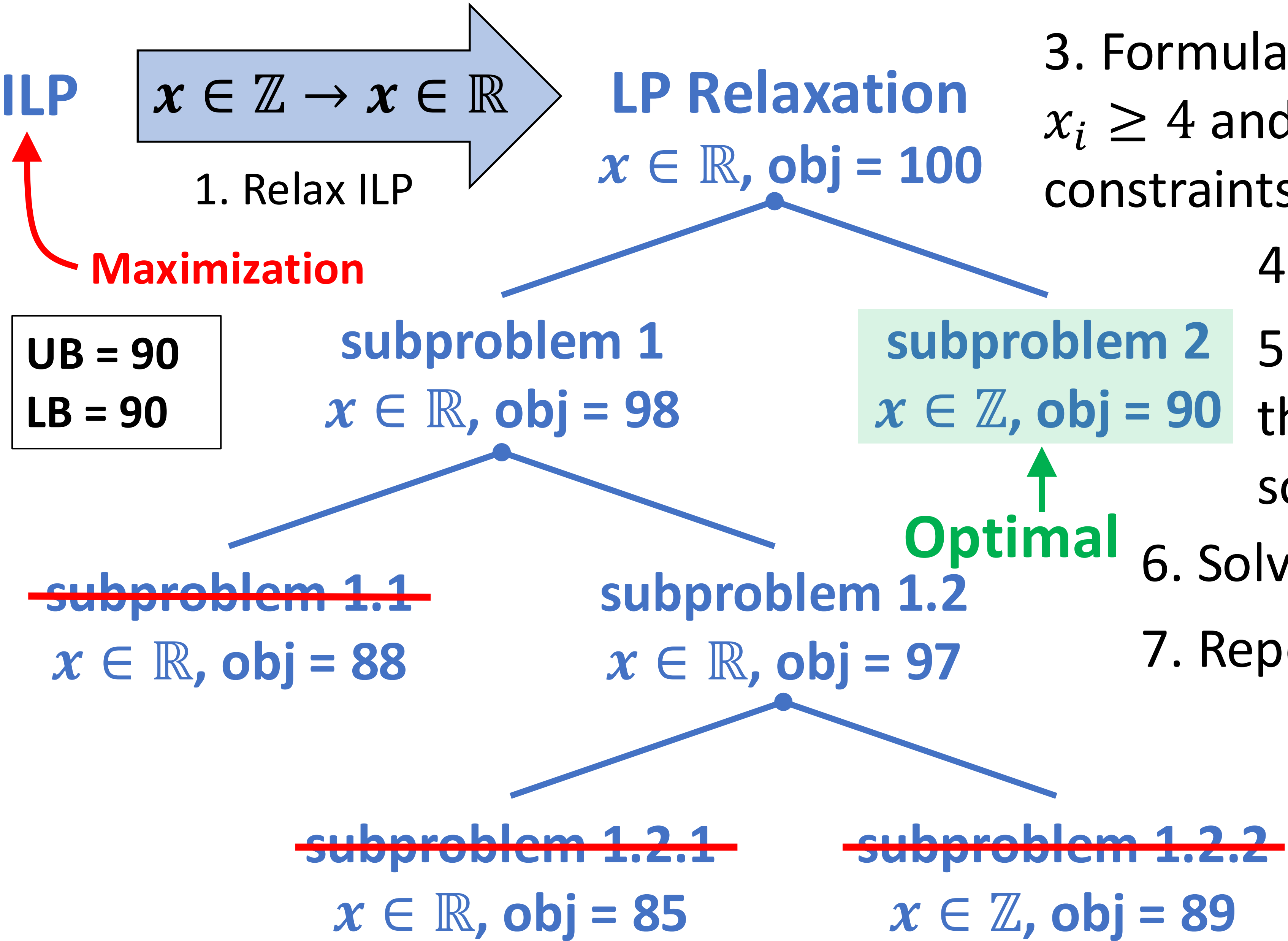
3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.



Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

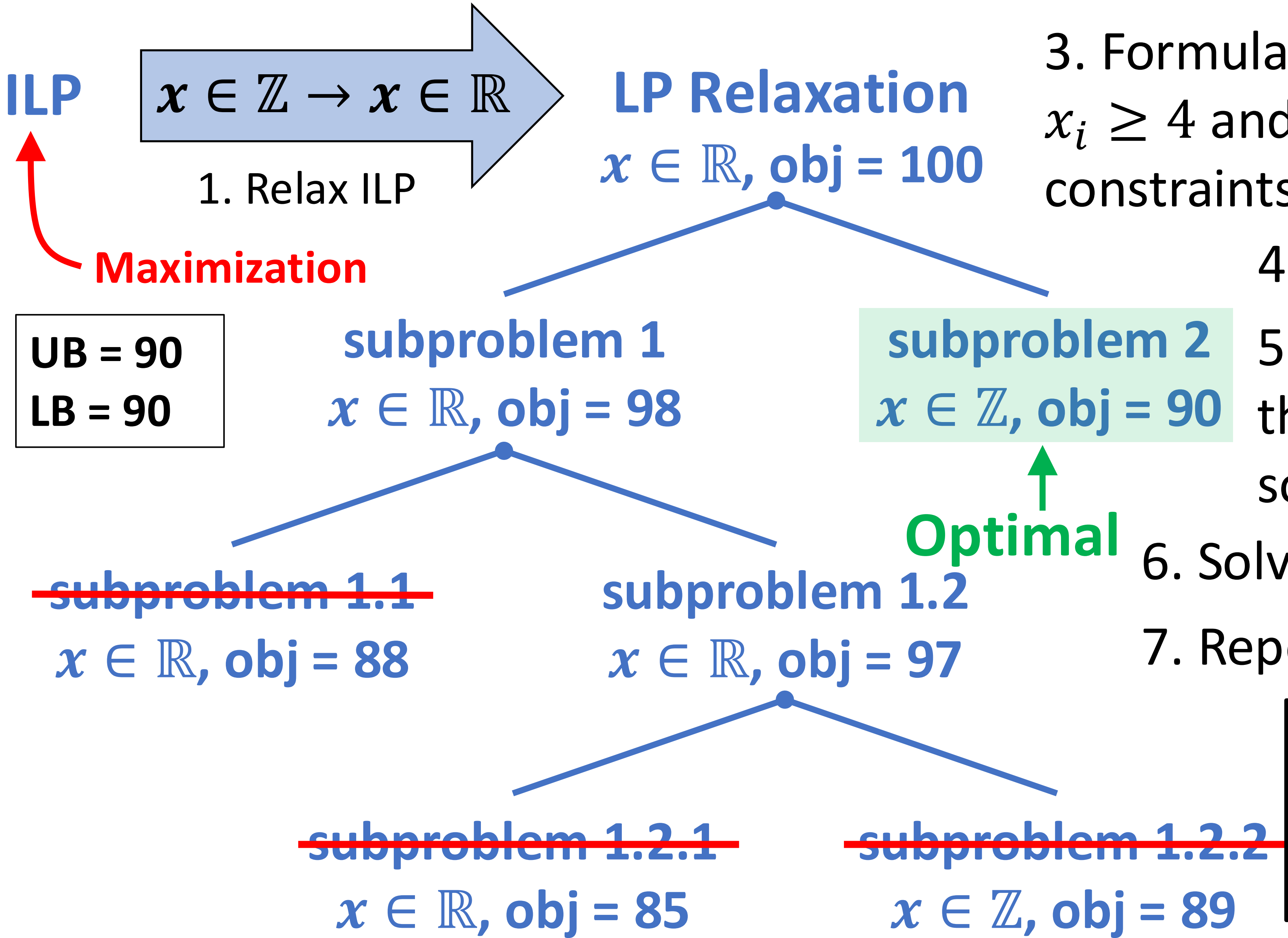
3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.



Is this process guaranteed to eventually find the optimal integer solution?

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

$\boldsymbol{x} \in \mathbb{R}$, **obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

Maximizatio

...blem LPs.

...o new LPs

...bproblem on

...er variable.

...em LPs.

UB = 90

LB = 90

~~subproblem~~

$\boldsymbol{x} \in \mathbb{R}$, **obj =**

Is this process guaranteed to eventually find the optimal integer solution?

Yes, given enough constraints like $x_i \geq 3$ and $x_i \leq 3$, all variables will be bounded to their optimal integer values.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

ILP $\boldsymbol{x} \in \mathbb{Z} \rightarrow \boldsymbol{x} \in \mathbb{R}$ **LP Relaxation**

1. Relax ILP

$\boldsymbol{x} \in \mathbb{R}$**, obj = 100**

3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

Maximizatio

...blem LPs.

...o new LPs

...bproblem on

...er variable.

...em LPs.

UB = 90

LB = 90

~~**subproblem**~~

$\boldsymbol{x} \in \mathbb{R}$**, obj =**

Is this process guaranteed to eventually find the optimal integer solution?

Yes, given enough constraints like $x_i \geq 3$ and $x_i \leq 3$, all variables will be bounded to their optimal integer values.

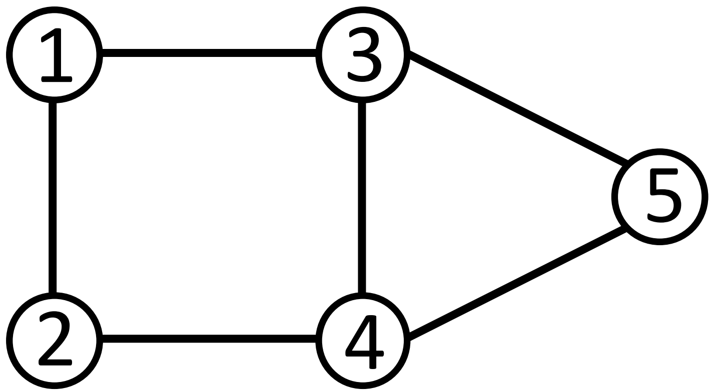
Other techniques (cutting planes, intelligently picking which x_i to split on) can help speed up the process.

Example

$x_i \in \{0,1\}$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$

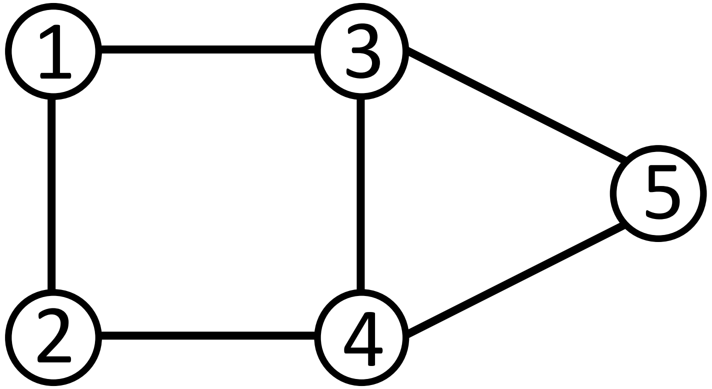


Example

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$



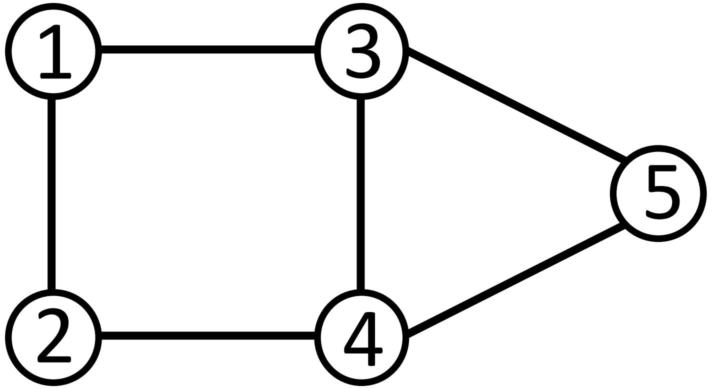
Example

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$

Obj = 2.5
$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$



Example

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$

Obj = 2.5

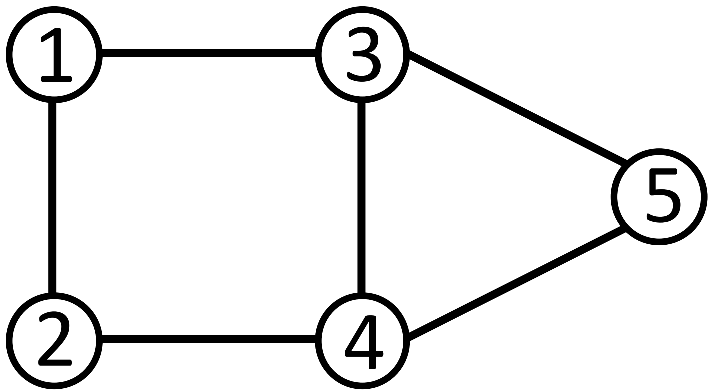
$x_1 = 0.5$

$x_2 = 0.5$

$x_3 = 0.5$

$x_4 = 0.5$

$x_5 = 0.5$



$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$
$$x_3 \leq 0$$

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$
$$x_3 \geq 1$$

Example

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to: $x_1 + x_2 \geq 1$

$x_1 + x_3 \geq 1$

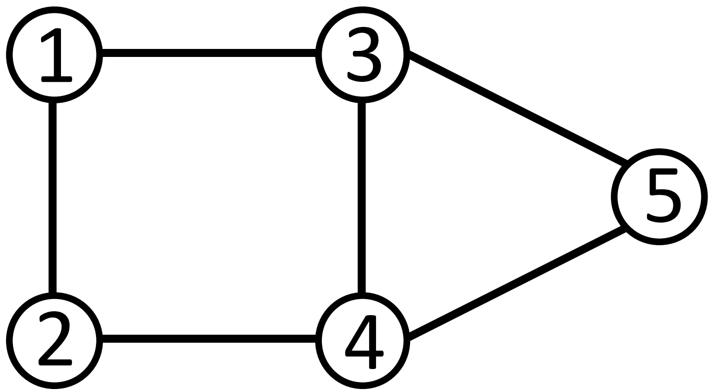
$x_2 + x_4 \geq 1$

$x_3 + x_4 \geq 1$

$x_3 + x_5 \geq 1$

$x_4 + x_5 \geq 1$

Obj = 2.5
$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$



$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to: $x_1 + x_2 \geq 1$

$x_1 + x_3 \geq 1$

$x_2 + x_4 \geq 1$

$x_3 + x_4 \geq 1$

$x_3 + x_5 \geq 1$

$x_4 + x_5 \geq 1$

$x_3 \leq 0$

Obj = ?
$x_1 =$
$x_2 =$
$x_3 = 0$
$x_4 =$
$x_5 =$

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to: $x_1 + x_2 \geq 1$

$x_1 + x_3 \geq 1$

$x_2 + x_4 \geq 1$

$x_3 + x_4 \geq 1$

$x_3 + x_5 \geq 1$

$x_4 + x_5 \geq 1$

$x_3 \geq 1$

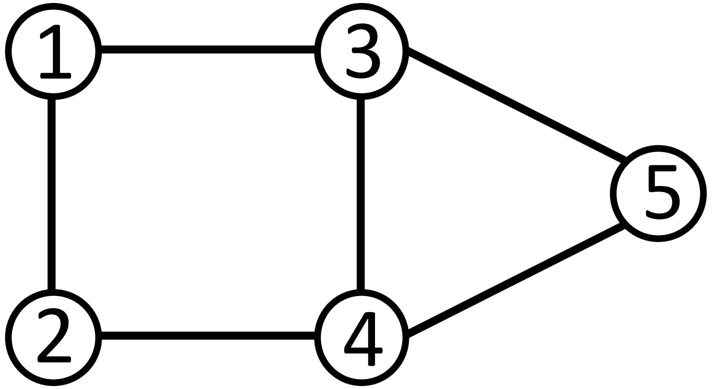
Example

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$x_1 + x_2 \geq 1$
$x_1 + x_3 \geq 1$
$x_2 + x_4 \geq 1$
$x_3 + x_4 \geq 1$
$x_3 + x_5 \geq 1$
$x_4 + x_5 \geq 1$

Obj = 2.5
$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$



$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$x_1 + x_2 \geq 1$
$x_1 + x_3 \geq 1$
$x_2 + x_4 \geq 1$
$x_3 + x_4 \geq 1$
$x_3 + x_5 \geq 1$
$x_4 + x_5 \geq 1$
$x_3 \leq 0$

Obj = ?
$x_1 = 1$
$x_2 =$
$x_3 = 0$
$x_4 = 1$
$x_5 = 1$

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$x_1 + x_2 \geq 1$
$x_1 + x_3 \geq 1$
$x_2 + x_4 \geq 1$
$x_3 + x_4 \geq 1$
$x_3 + x_5 \geq 1$
$x_4 + x_5 \geq 1$
$x_3 \geq 1$

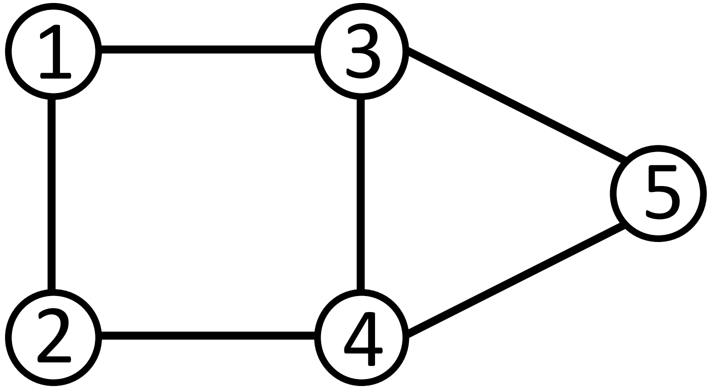
Example

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$

Obj = 2.5
$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$



$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$
$$x_3 \leq 0$$

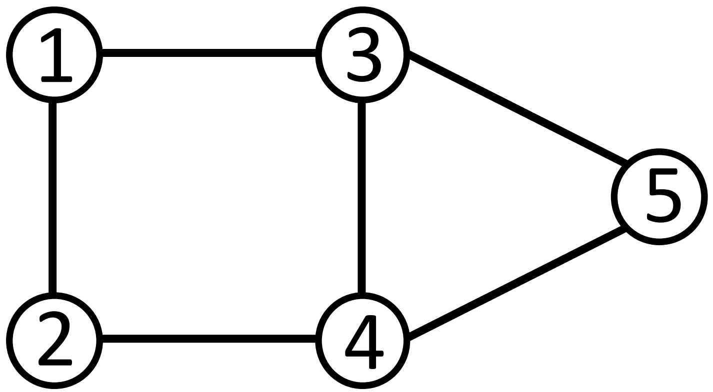
Obj = 3
$x_1 = 1$
$x_2 = 0$
$x_3 = 0$
$x_4 = 1$
$x_5 = 1$

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$
$$x_3 \geq 1$$

Example



$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$

Obj = 2.5
$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$
$$x_3 \leq 0$$

Obj = 3
$x_1 = 1$
$x_2 = 0$
$x_3 = 0$
$x_4 = 1$
$x_5 = 1$

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_3 + x_5 \geq 1$$
$$x_4 + x_5 \geq 1$$
$$x_3 \geq 1$$

Obj = ?
$x_1 =$
$x_2 =$
$x_3 = 1$
$x_4 =$
$x_5 =$

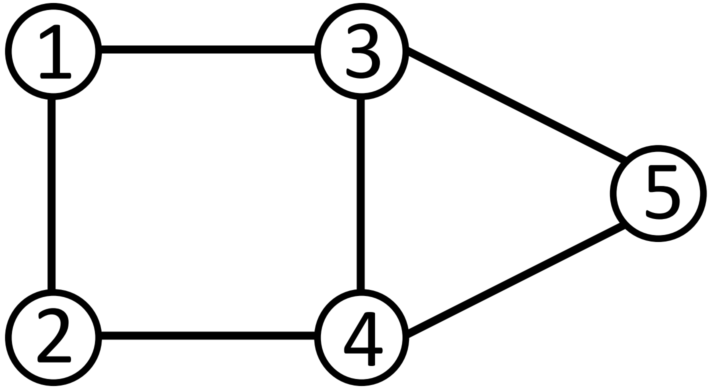
Example

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$x_1 + x_2 \geq 1$
$x_1 + x_3 \geq 1$
$x_2 + x_4 \geq 1$
$x_3 + x_4 \geq 1$
$x_3 + x_5 \geq 1$
$x_4 + x_5 \geq 1$

Obj = 2.5
$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 0.5$
$x_4 = 0.5$
$x_5 = 0.5$



$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$x_1 + x_2 \geq 1$
$x_1 + x_3 \geq 1$
$x_2 + x_4 \geq 1$
$x_3 + x_4 \geq 1$
$x_3 + x_5 \geq 1$
$x_4 + x_5 \geq 1$
$x_3 \leq 0$

Obj = 3
$x_1 = 1$
$x_2 = 0$
$x_3 = 0$
$x_4 = 1$
$x_5 = 1$

$x_i \in [0,1]$

Objective: $\min x_1 + x_2 + x_3 + x_4 + x_5$

Subject to:
$x_1 + x_2 \geq 1$
$x_1 + x_3 \geq 1$
$x_2 + x_4 \geq 1$
$x_3 + x_4 \geq 1$
$x_3 + x_5 \geq 1$
$x_4 + x_5 \geq 1$
$x_3 \geq 1$

Obj = 3
$x_1 = 0.5$
$x_2 = 0.5$
$x_3 = 1$
$x_4 = 0.5$
$x_5 = 0.5$

Solving ILPs

$x_1, x_2 \in \mathbb{Z}$

Objective: $\min x_1 + x_2$

Subject to:
$$x_1 \leq 1$$
$$x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

Can we optimally solve this ILP?

Solving ILPs

$x_1, x_2 \in \mathbb{Z}$

Objective: $\min x_1 - x_2$

Subject to: $x_1 \leq 1$

$x_2 \leq 1$

$x_1, x_2 \geq 0$

Can we optimally solve this ILP?

Solving ILPs

$x_1, x_2 \in \mathbb{Z}$

Objective: $\min -x_1 - x_2$

Subject to:
$$x_1 \leq 1$$
$$x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

Can we optimally solve this ILP?

Solving ILPs

$x_1, x_2 \in \mathbb{Z}$

Objective: $\min -\frac{\pi}{\sqrt{2}} x_1 + \frac{\log 2}{e} x_2 + \frac{32}{17}$

Subject to:
$$x_1 \leq 1$$
$$x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

Can we optimally solve this ILP?

Solving ILPs

$x_1, x_2 \in \mathbb{Z}$

Objective: $\min -\frac{\pi}{\sqrt{2}} x_1 + \frac{\log 2}{e} x_2 + \frac{32}{17}$

Subject to: $x_1 \leq 1$

$x_2 \leq 1$

$x_1, x_2 \geq 0$

Can we optimally solve this ILP?

Why?

Solving ILPs

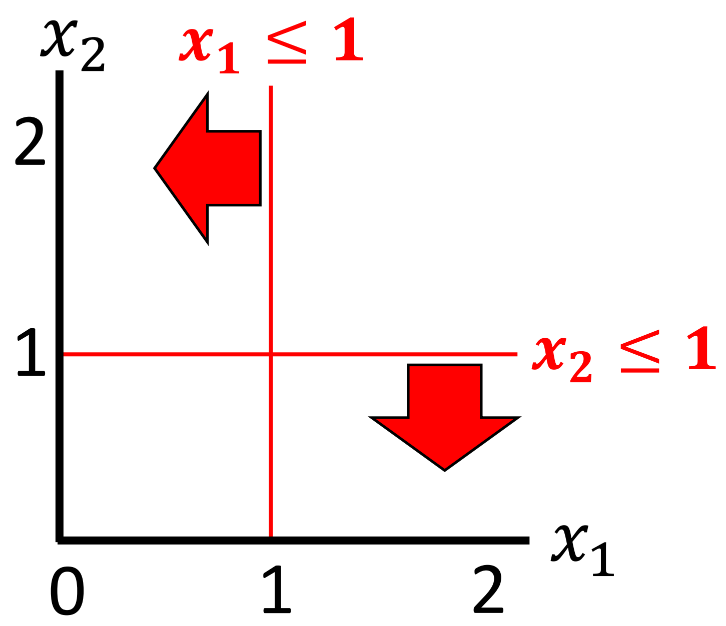
$x_1, x_2 \in \mathbb{Z}$

Objective: $\min -\frac{\pi}{\sqrt{2}} x_1 + \frac{\log 2}{e} x_2 + \frac{32}{17}$

Subject to: $x_1 \leq 1$

$x_2 \leq 1$

$x_1, x_2 \geq 0$



Can we optimally solve this ILP?

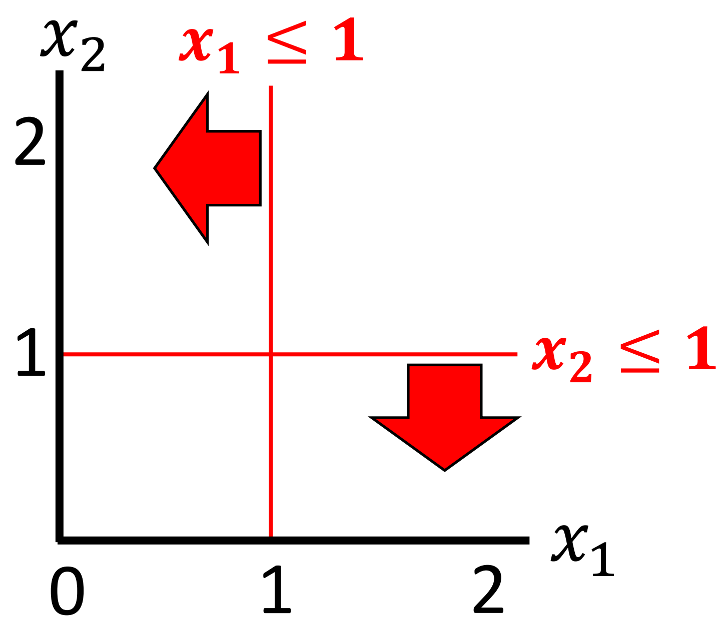
Why?

Solving ILPs

$x_1, x_2 \in \mathbb{Z}$

Objective: $\min -\frac{\pi}{\sqrt{2}} x_1 + \frac{\log 2}{e} x_2 + \frac{32}{17}$

Subject to:
$x_1 \leq 1$
$x_2 \leq 1$
$x_1, x_2 \geq 0$



Can we optimally solve this ILP?

Why?

If all feasible region vertices are integer-valued, an optimal solution is integer-valued, regardless of the objective function.

Vertex Cover

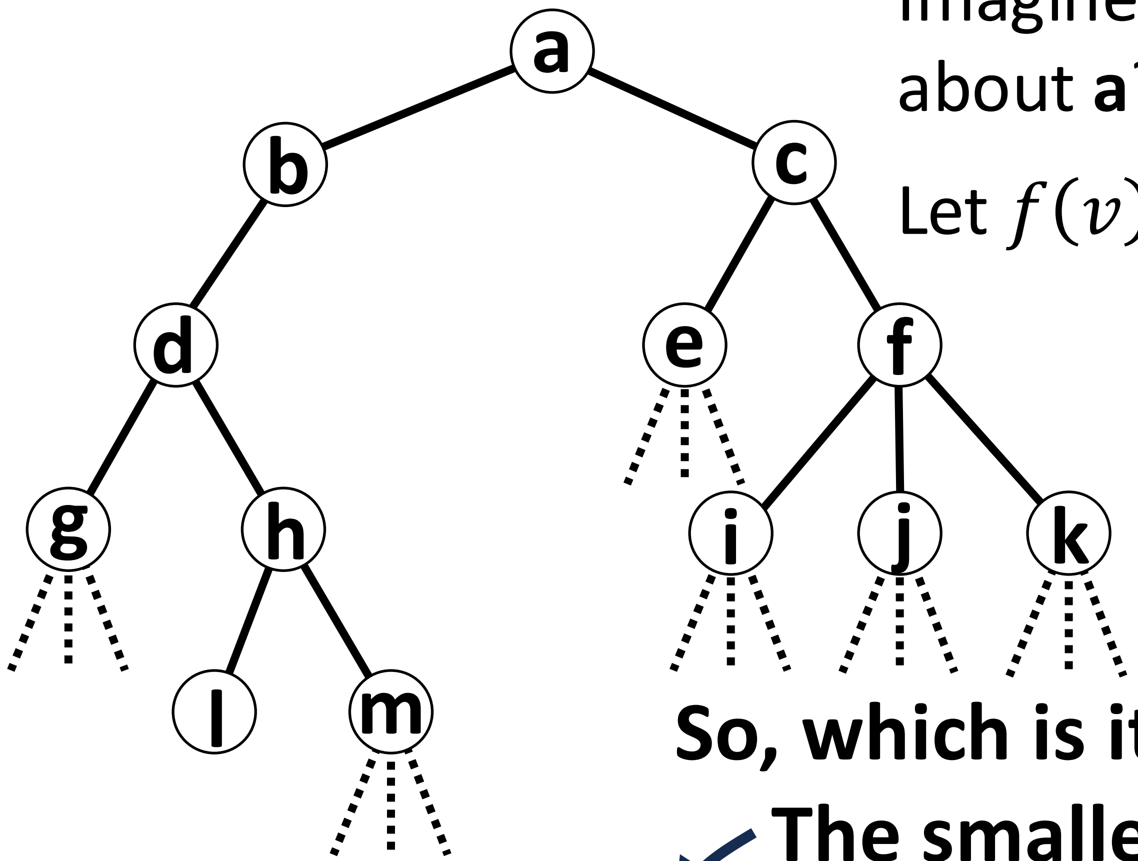
$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

General Graphs: NP-Hard

Vertex Cover in Trees



Imagine the minimum vertex cover. What can we say about **a**? **a** is in a minimum vertex cover, or it's not.

Let $f(v) =$ Size of minimum vertex cover rooted at v.

If a **is** in a minimum VC

$$f(a) = 1 + f(b) + f(c)$$

If a **is not** in a minimum VC

$$f(a) = 2 + f(d) + f(e) + f(f)$$

So, which is it?

The smallest!

$$f(v) = \min\left\{1 + \sum_{u \in \text{child}(v)} f(u),\ |\text{child}(v)| + \sum_{w \in \text{grandchild}(v)} f(w)\right\}$$

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i,j)$

General Graphs: NP-Hard

Trees: $O(n)$

Vertex Cover

$x_i \in \{0,1\} =$ Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

But they all use the same ILP!

General Graphs: NP-Hard

Trees: $O(n)$

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

But they all use the same ILP!

General Graphs: NP-Hard

Trees: $O(n)$

Goal: Find some property of special ILPs that give them integer-vertexed feasible regions.

Total Unimodularity

Definition: A matrix is totally unimodular if the determinant of any square submatrix of it is 0, 1, or -1.

Total Unimodularity

Definition: A matrix is totally unimodular if the determinant of any square submatrix of it is 0, 1, or −1.

This implies that totally unimodular matrices are composed of only 0's, 1's, and -1's, since single elements are square submatrices.

Total Unimodularity

Definition: A matrix is totally unimodular if the determinant of any square submatrix of it is 0, 1, or −1.

This implies that totally unimodular matrices are composed of only 0's, 1's, and -1's, since single elements are square submatrices.

It also means that if A is totally unimodular, A^{T} is too, since $\det(B) = \det(B^{\mathsf{T}})$.

Total Unimodularity

Definition: A matrix is totally unimodular if the determinant of any square submatrix of it is 0, 1, or −1.

Theorem (Hoffman-Kruskal, 1956):

$A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ The feasible region: $\{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

Total Unimodularity

Definition: A matrix is totally unimodular if the determinant of any square submatrix of it is 0, 1, or −1.

Theorem (Hoffman-Kruskal, 1956):

$A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ The feasible region: $\{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

Objective:	max $c^T x$
Subject to:	$A x \leq b$
	$x \geq 0$

If A is totally unimodular, the ILP can be solved in polynomial time!

Total Unimodularity

Definition: A matrix is <u>totally unimodular</u> if the determinant of any square submatrix of it is 0, 1, or −1.

Theorem (Hoffman-Kruskal, 1956):

$A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ The feasible region: $\{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

$$\begin{pmatrix} 4 & 5 & 7 & 3 & 0 & 1 \\ 1 & 2 & 8 & 4 & 6 & 5 \\ 9 & 9 & 1 & 5 & 7 & 3 \\ 9 & 5 & 6 & 6 & 8 & 2 \\ 1 & 1 & 0 & 1 & 2 & 7 \end{pmatrix} \Rightarrow \begin{bmatrix} 2 & 8 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix}$$

Total Unimodularity

Definition: A matrix is <u>totally unimodular</u> if the determinant of any square submatrix of it is 0, 1, or −1.

Theorem (Hoffman-Kruskal, 1956):

$A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ The feasible region: $\{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

Theorem (Ghouila-Houri, 1962):

$A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ For every subset of rows R, there is a partition $R = R_1 \cup R_2$ such that for every column j,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1,0,1\}$$

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Vertex Cover

$x_i \in \{0,1\} =$ Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i,j)$

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Theorem (Ghouila-Houri, 1962):

$A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ For every subset of rows R, there is a partition $R = R_1 \cup R_2$ such that for every column j,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1,0,1\}$$

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i,j)$

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Theorem (Ghouila-Houri, 1962):

$A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ For every subset of rows R, there is a partition $R = R_1 \cup R_2$ such that for every column j,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1,0,1\}$$

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

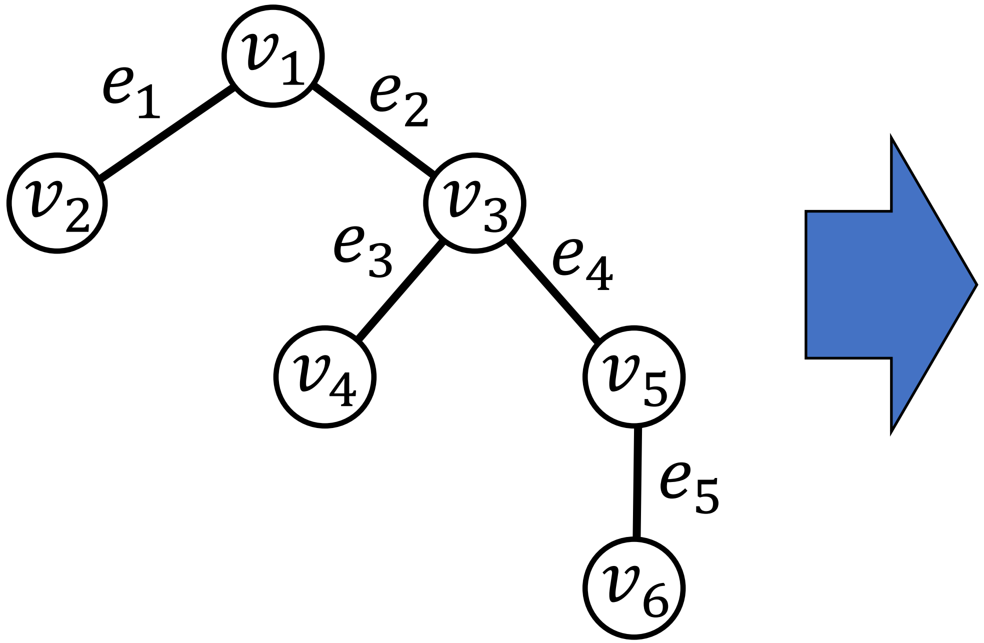
Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i,j)$

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$



$A =$??

Vertex Cover

$x_i \in \{0,1\} =$ Indicates if vertex i is selected.

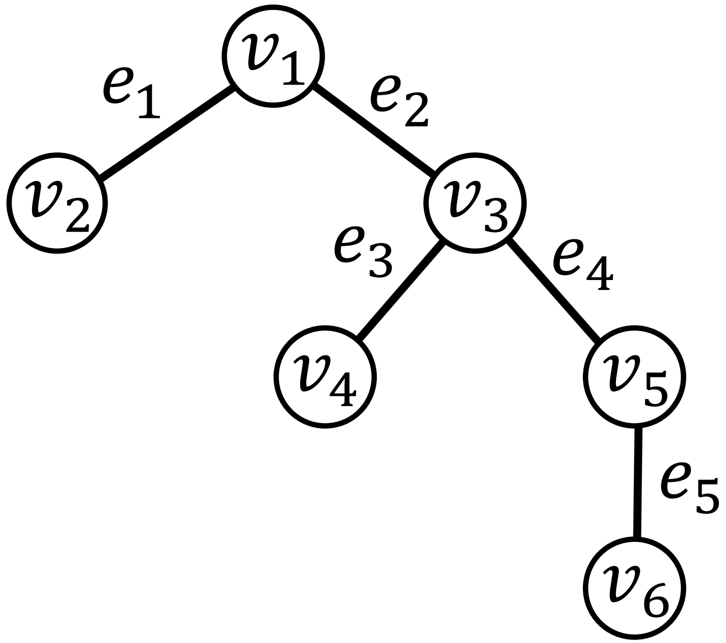
Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i,j)$

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$



$A =$

	x_1	x_2	x_3	x_4	x_5	x_6
e_1						
e_2						
e_3						
e_4						
e_5						

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

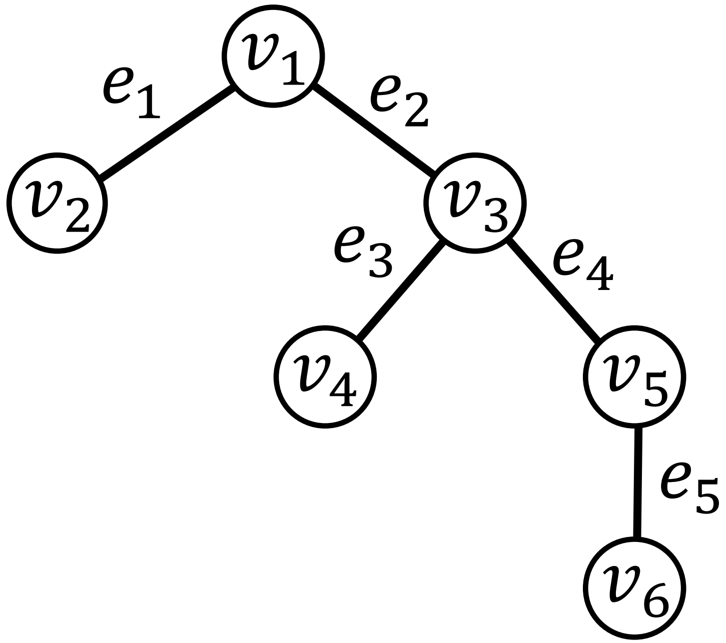
Objective: $\min \sum_i x_i$

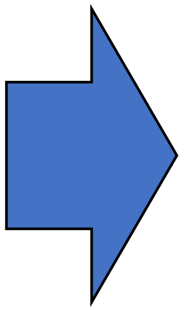
Subject to: $x_i + x_j \geq 1$, for each edge $e = (i,j)$

Objective: max $c^T x$

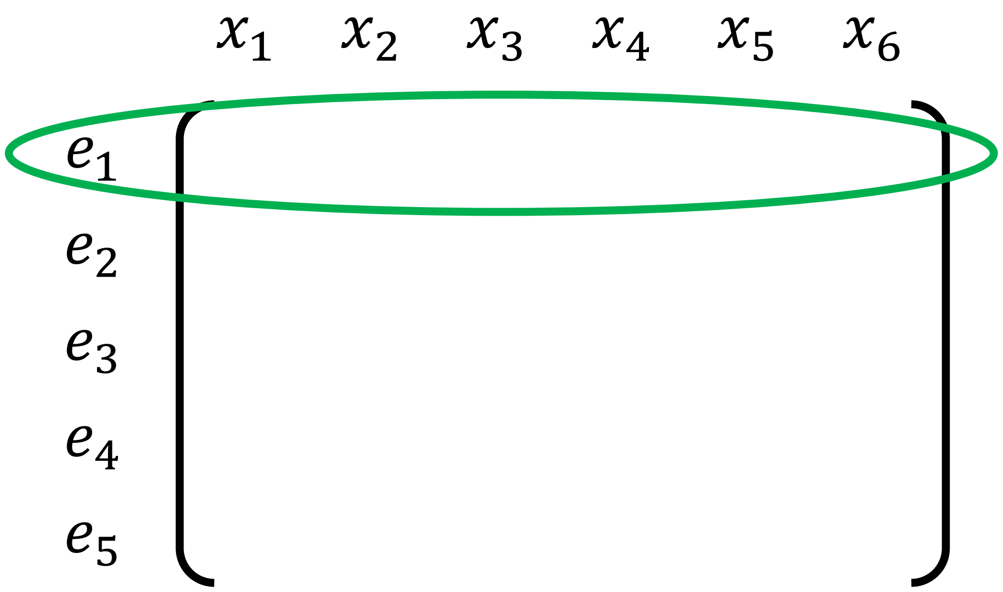
Subject to: $A x \leq b$

$x \geq 0$





$A =$



Vertex Cover

$x_i \in \{0,1\} =$ Indicates if vertex i is selected.

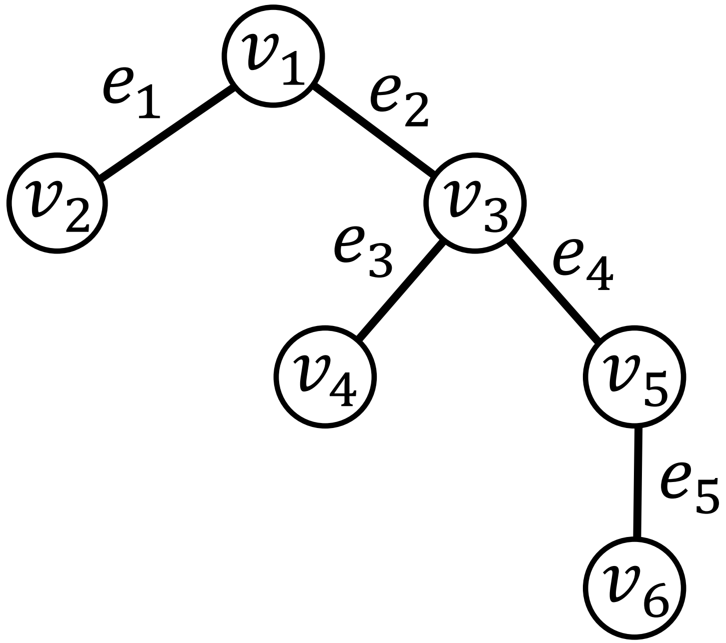
Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$



$$A = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array}$$

Vertex Cover

$x_i \in \{0,1\} =$ Indicates if vertex i is selected.

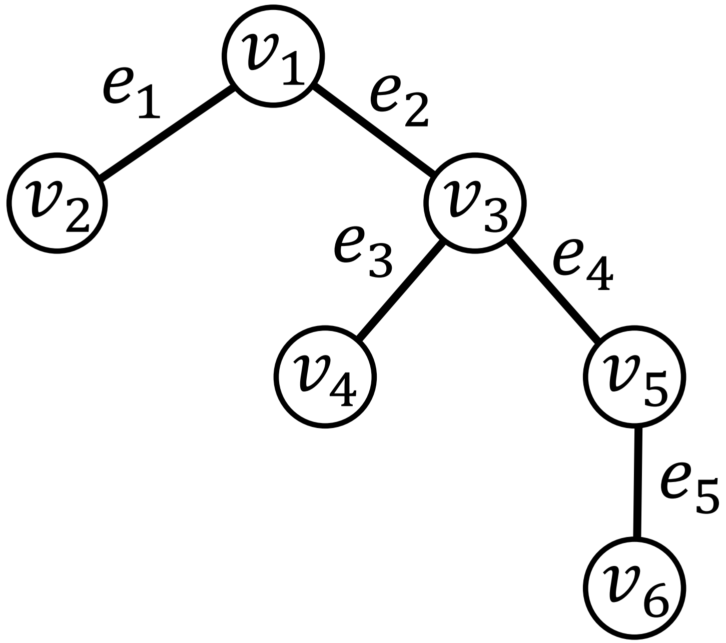
Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$



$A =$

	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	1	0	0	0	0
e_2	1	0	1	0	0	0
e_3						
e_4						
e_5						

Vertex Cover

$x_i \in \{0,1\} =$ Indicates if vertex i is selected.

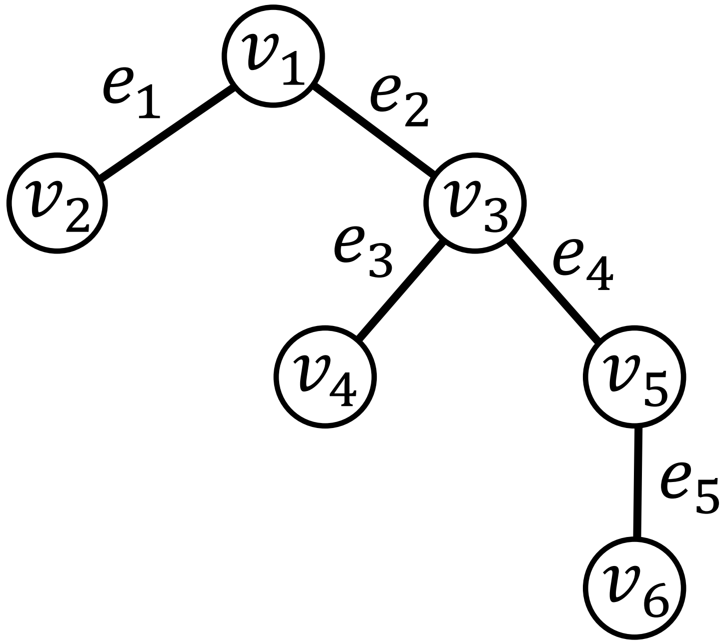
Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Objective: max $c^T x$

Subject to: $A x \leq b$

$x \geq 0$



$A =$

	x_1	x_2	x_3	x_4	x_5	x_6
e_1	1	1	0	0	0	0
e_2	1	0	1	0	0	0
e_3	0	0	1	1	0	0
e_4	0	0	1	0	1	0
e_5	0	0	0	0	1	1

Vertex Cover

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ For every subset of rows R, there is a partition $R = R_1 \cup R_2$ such that for every column j,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1,0,1\}$$

$$A = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Total Unimodularity

Definition: A matrix is totally unimodular if the determinant of any square submatrix of it is 0, 1, or −1.

This implies that totally unimodular matrices are composed of only 0's, 1's, and -1's, since single elements are square submatrices.

It also means that if A is totally unimodular, A^{T} is too, since $\det(B) = \det(B^{\mathsf{T}})$.

Vertex Cover

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ For every subset of rows R, there is a partition $R = R_1 \cup R_2$ such that for every column j,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1,0,1\}$$

$$A = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$A^{\mathsf{T}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Vertex Cover

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular $\Leftrightarrow$ For every subset of rows R, there is a partition $R = R_1 \cup R_2$ such that for every column j,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

$$A^\mathsf{T} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Vertex Cover

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m\times n}$ is totally unimodular $\Leftrightarrow$ For every subset of rows R, there is a partition $R = R_1 \cup R_2$ such that for every column j,

$$\sum_{i\in R_1} A_{ij} - \sum_{i\in R_2} A_{ij} \in \{-1,0,1\}$$

$$A^{\mathsf{T}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Label alternating generations of vertices into two sets B and G.